

ON THE EXISTENCE OF SLE TRACE: FINITE ENERGY DRIVERS AND NON-CONSTANT κ

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ABSTRACT. Existence of Loewner trace is revisited. We identify finite energy paths (the “skeleton of Wiener measure”) as natural class of regular drivers for which we find simple and natural estimates in terms of their (Cameron–Martin) norm. Secondly, now dealing with potentially rough drivers, a representation of the derivative of the (inverse of the) Loewner flow is given in terms of a rough- and then pathwise Föllmer integral. Assuming the driver within a class of Itô-processes, an exponential martingale argument implies existence of trace. In contrast to classical (exact) SLE computations, our arguments are well adapted to perturbations, such as non-constant κ (assuming < 2 for technical reasons) and additional finite-energy drift terms.

1. INTRODUCTION

Classical theory of Loewner evolution gives a one-to-one correspondence between scalar continuous drivers (no smoothness assumptions) and families of continuously growing compact sets in the complex upper half-plane \mathbb{H} . There is much interest in the case where these sets admit a continuous trace (or even better: are given by a simple curve in \mathbb{H}). The famous *Rohde–Schramm theorem* [RS05] asserts that Brownian motion with diffusivity $\kappa \neq 8$ a.s. gives rise to a continuous trace (simple when $\kappa \leq 4$), better known as SLE(κ)-curves.¹ The trace also exists for SLE(8) but the proof follows indirectly from the convergence of uniform spanning tree to SLE(8). (We note that the proofs are probabilistic in nature – ultimately an application of the Borel–Cantelli lemma – and gives little insight about the exceptional set.) Deterministic aspects were subsequently explored by Marshall, Rohde, Lind, Huy Tran, Johansson Viklund and others (see e.g. [Joh15] and the references therein). We observe a number of similarities between the *Itô map*, which takes a Brownian driver to a diffusion path) with the *Schramm–Löwner map* which takes a Brownian driver to SLE trace γ (a “rough”, in the sense of non-smooth, path in \mathbb{H}),

$$\Phi_{SL} : \sqrt{\kappa}B(\omega) \mapsto \gamma(\omega).$$

In both cases, there is a “Young” regime (case of drivers with Hölder exponent better than $1/2$) in which case one can fully rely on deterministic theory.² Also, in both the cases, Brownian motion does not fall in the afore-mentioned “Young” regime, and yet both the Itô- resp. (Schramm-)Löwner map are well-defined measurable maps. While the Itô map, also thanks to rough path theory, is now very well understood, the afore-mentioned proof of the Rohde–Schramm theorem - despite being state of art - is not fully satisfactory. For instance, the case $\kappa = 8$ still resists a direct analysis.

¹We shall make no attempt here to review the fundamental importance of SLE theory within probability and statistical mechanics. See e.g. [Law08] and the references therein.

²The analogy is not perfect: Young differential equations of form $dY = f(Y)dX$ are invariant under reparametrization, hence most naturally formulated in a p -variation, $p < 2$, context, whereas Loewner evolution is classically tied to parametrization by half-plane capacity.

Even robustness in the parameter κ turns out to be a decisively non-trivial issue, only recently settled in [JVRW14] under the (technical) restriction of $\kappa < 8(2 - \sqrt{3})$.

To some extent, the “pathwise” theory of Loewner evolutions, concerning existence of trace, has been settled by Rohde–Schramm in form of the following

Theorem 1. [RS05] *Loewner evolution with driver U , a continuous real valued path with $U_0 = 0$, admits a continuous trace if and only if*

$$\gamma_t := \lim_{y \rightarrow 0+} f_t(iy + U_t)$$

exists and is continuous in t . In this case, γ is the trace.

In the above theorem, following standard notation, $f_t(z) = g_t^{-1}(z)$ and for each $z \in \bar{\mathbb{H}} \setminus 0$, let $g_t(z)$ denote the solution of the LDE (Loewner’s differential equation)

$$(1.1) \quad \dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Evenso, it is a non-trivial matter, and the essence of the afore-mentioned Rohde–Schramm theorem, to see that this applies to a.e. Brownian sample path. It is here that one has to work with Whitney-type boxes and a subtle Borel–Cantelli argument which in fact misses the case $\kappa = 8$, subsequently handled with different methods. Readers familiar with the details of the proof may observe that a harmless finite-energy perturbation of the driver will already cause some serious complications, whereas it is a priori clear from the Cameron–Martin theorem that SLE driven by $\sqrt{\kappa}B + h$, where h is \int_0^\cdot of some L^2 -function, does produce a continuous trace. (Such perturbations are relatively harmless for the Itô-map, essentially because integration against $dh = \dot{h}dt$ is deterministic and SDEs driven by $B + h$ can be dealt with via flow decompositions.)

In fact, we observed with some surprise that Loewner evolution driven by finite-energy paths, despite being the “skeleton” of Wiener-measure, has not been analyzed. With regard to Lind’s “1/2-Hölder norm < 4 ” condition, we note that a finite-energy path h is indeed 1/2-Hölder (by a simple Sobolev embedding), but may have arbitrarily large 1/2-Hölder norm. Evenso, there is a “poor man’s argument” that allows to see that such h generate a simple curve trace: the remark is that h is *vanishing* 1/2-Hölder in the sense $\sup_{s,t: |t-s| < \varepsilon} \frac{|h_t - h_s|}{|t-s|^{1/2}} \rightarrow 0$ with $\varepsilon \rightarrow 0$, so that γ is given by suitable concatenation of (conformally deformed) simple curves. A better understanding of the situation is given by Theorem 2 below. To state it define \mathcal{H}_t as the space of absolutely continuous $h : [0, t] \rightarrow \mathbb{R}$, with finite energy i.e. square-integrable derivative \dot{h} , and norm-square

$$||h||_t^2 := \int_0^t |\dot{h}_s|^2 ds.$$

Also, C_T^α is the space of paths defined $[0, T]$, with Hölder exponent $\alpha \in (0, 1]$. Write C_T^{p-var} for the space of continuous paths on $[0, T]$ of finite p -variation; note that α -Hölder implies finite $1/\alpha$ -variation. At last, $C_{0,T}^\alpha$ indicates paths on $[0, T]$ which are started at 0. For instance, given a standard Brownian motion B , with probability one $\sqrt{\kappa}B(\omega) \in C_{0,T}^\alpha$ for any $\alpha < 1/2$.

Theorem 2. *Let $T > 0$ and $U \in \mathcal{H}_T$.*

(i) *The following estimate holds for all $y > 0$ and $t \in [0, T]$,*

$$(1.2) \quad |f'_t(iy + U_t)| \leq \exp \left[\frac{1}{4} \|U\|_t^2 \right].$$

(ii) *The Loewner-trace $\gamma =: \Phi_{SL}(U)$ exists and is a simple curve.*

(iii) *The trace is uniformly $1/2$ -Hölder in the sense that, for some constant C ,*

$$(1.3) \quad \|\gamma\|_{1/2} \leq C \exp \left[C \|U\|_T^2 \right]$$

(iv) *The map $t \mapsto \gamma(t^2)$ is Lipschitz continuous on $[0, T]$. As a consequence, the trace is of bounded variation and Lipschitz away from $0+$.*

(v) *On bounded sets in \mathcal{H}_T , the Schramm–Loewner map is continuous from $C[0, T]$ to $C^{1/2-\epsilon}([0, T], \mathbb{H})$, any $\epsilon > 0$.*

(vi) *The Schramm–Loewner map is continuous from \mathcal{H}_T to $C^{(1+\epsilon)-var}$, any $\epsilon > 0$.*

Our second contribution is a *pathwise* inequality that is well-suited to obtain existence of trace for stochastic drivers beyond Brownian motion. To state it, let us say that $U : [0, T] \rightarrow \mathbb{R}$ has *finite quadratic-variation in sense of Föllmer* if (along some *fixed* sequence of partitions $\pi = (\pi_n)$ of $[0, T]$, with mesh-size going to zero)

$$\exists \lim_{n \rightarrow \infty} \sum_{[r,s] \in \pi_n} (U_{s \wedge t} - U_{r \wedge t})^2 =: [U]_t^\pi$$

and defines a continuous map $t \mapsto [U]_t^\pi \equiv [U]_t$. A function V on $[0, T]$ is called *Föllmer-Itô integrable* (against U , along π) if

$$\exists \lim_{n \rightarrow \infty} \sum_{[r,s] \in \pi_n} V_s (U_{s \wedge t} - U_{r \wedge t}) =: \int_0^t V d^\pi U.$$

(Föllmer [Foe81] shows that integrands of gradient form are integrable in this sense and so defines pathwise integrals of the form $\int \nabla F(U) d^\pi U$.) If the bracket is furthermore Lipschitz, in the sense that

$$(1.4) \quad \sup_{0 \leq s < t \leq T} \frac{|[U]_t - [U]_s|}{t - s} \leq \kappa < \infty,$$

write $U \in \mathcal{Q}_T^{\pi, \kappa}$. For instance, whenever π is nested, a martingale argument shows that $\sqrt{\kappa}B(\omega) \in \mathcal{Q}_T^{\pi, \kappa}$ with probability one. We insist again that the following result is entirely deterministic and highlights the role of the (pathwise) property (1.4) relative to existence of $\text{SLE}(\kappa)$ trace.

Theorem 3. *Let $T > 0$ and $U \in C_{0,T}^\alpha \cap \mathcal{Q}_T^{\pi, \kappa}$ for some $1/3 < \alpha < 1/2$ and $\kappa < 2$. For fixed $t \in [0, T]$ set $\beta_s := U_t - U_{t-s}$ and then, for arbitrarily chosen $A \in \mathcal{H}_t$, consider the decomposition³*

$$\beta = N + A.$$

Then there exists a continuous function \dot{G} , Föllmer integrable against N , so that for some $b > 2, p > 1$ and $\epsilon > 0$, depending only on κ , we have, for all $y > 0$ and $t \in [0, T]$,

$$(1.5) \quad |f'_t(iy + U_t)|^b \leq \exp \left(b \int_0^t \dot{G}_r d^\pi N_r - \frac{pb^2}{2} \int_0^t \dot{G}_r^2 d[N]_r^\pi \right) \exp \left(\frac{b}{4\epsilon} \|A\|_t^2 \right).$$

³One could write $\beta^{(t)} = N^{(t)} + A^{(t)}$ to emphasize dependence on t .

Several remarks are in order.

Remark 1. $U \in \mathcal{Q}_T^{\pi, \kappa}$ iff $N \in \mathcal{Q}_T^{\pi, \kappa}$ since $[N]_t = [\beta]_t = [U]_T - [U]_{T-t}$.

Remark 2. An explicit form of \dot{G} is found in (2.4). Remark that \dot{G}_s is obtained as function of $(\beta_u : 0 \leq u \leq s)$, and in fact is controlled by β in the sense of Gubinelli [Gub04] or [FH14, Ch. 4], which is a technical aspect in the proof.

Remark 3. We believe the restriction $\kappa < 2$ to be of technical nature.

Write $C_{0,T}^{w,1/2}$ for “weakly” 1/2-Hölder paths on $[0, T]$, started at zero. Following [JVL11], this means a modulus of continuity of the form $\omega(r) = r^{1/2}\varphi(1/r)$ for a “subpower” function φ (that is, $\varphi(x) = o(x^\nu)$ for all $\nu > 0$, as $x \rightarrow \infty$). Thanks to Lévy’s modulus of continuity, with probability one, $\sqrt{\kappa}B(\omega) \in C_{0,T}^{w,1/2} \subset C_{0,T}^\alpha$ for any $\alpha < 1/2$. (A general Besov–Lévy modulus embedding appears e.g. in [FV10, p.576].)

Corollary 1. Let $T > 0$ and consider random $U = U(\omega)$ with $U(\omega) \in C_{0,T}^{w,1/2} \cap \mathcal{Q}_T^\kappa$ for $\kappa < 2$ a.s. For fixed $t \in [0, T]$, define β as before and assume β is a continuous semimartingale w.r.t. to some filtration, with canonical decomposition $\beta = N + A$ into local martingale N and bounded variation part $A \in \mathcal{H}_t$, so that $\|A\|_t^2$ has sufficiently high (depending only on κ) exponential moments finite uniform in t . Then the Loewner-trace $\gamma =: \Phi_{SL}(U)$ exists.

Proof. By assumption, we can apply Theorem 3 to a fixed realization of $U = U(\omega)$ in a set of full measure. Moreover, in view of the assumed semimartingale structure of $\beta = N + A$, our interpretation of the right-hand side of (1.5) can now be in classical Itô-sense. (In the semimartingale case, the Föllmer’s integral and quadratic variation, along suitable sequences of partitions, coincide with Itô’s notion.)

With $b > 2$ and then $p > 1, \epsilon > 0$ as in Theorem 3, let q be the Hölder conjugate of p . Hölder’s inequality gives

$$\mathbb{E}[|f'_t(iy + U_t)|^b] \leq \mathbb{E}\left[\mathcal{E}\left(pb \int_0^t \dot{G}dN\right)\right]^{\frac{1}{p}} \mathbb{E}\left[\exp\left(\frac{qb}{4\epsilon} \int_0^t \dot{A}_r^2 dr\right)\right]^{\frac{1}{q}}$$

where $\mathcal{E}(\dots)$ denotes the stochastic exponential. Since \dot{G} is adapted to β , its integral against N , is again a local martingale and so it the stochastic exponential. By positivity it is also a super-martingale, started at 1, and thus of expectation less equal one. Hence, for $b > 2$, have $\mathbb{E}[|f'_t(iy + U_t)|^b] < \infty$, uniformly in $t \in [0, T]$ and $y > 0$. Together with $U \in C_{0,T}^{w,1/2}$ a.s. this is well-known (cf. appendix) to imply existence of trace. \square

Again, some remarks are in order.

Remark 4. We cannot make a semimartingale assumption for the Loewner driver U since the time-reversal of a semimartingales can fail to be a semimartingale. That said, time-reversal of diffusion was studied by a number of authors including Millet, Nualart, Sanz, Pardoux ... and sufficient conditions on “diffusion Loewner drivers” could be given by tapping into this literature.

Remark 5. As revealed by the above proof of the corollary, the only purpose of the semimartingale assumption of β is to get good concentration of measure for $\int \dot{G}dN$. Recent progress on concentration of measure for pathwise stochastic integrals [FH14, Ch.11.2], also [Ch. 5] for some Gaussian examples of finite QV in Föllmer sense, suggest a possibility to study random Loewner evolutions without martingale methods.

Theorem 3 and its corollary have little new to say about existence of trace for SLE_κ , especially with its restriction $\kappa < 2$. However, it is capable of dealing with non-Brownian drivers, including situations with non-constant κ , and \mathcal{H} -perturbations thereof.

Example 1. (Classical SLE_κ) As a warmup, consider Loewner driver $U = \sqrt{\kappa}B$ with fixed $\kappa < 2$. Since, for fixed t , $\beta_s := U_t - U_{t-s}$ defines another Brownian motion, we can trivially decompose with $N = \beta$, $A \equiv 0$ and thus obtain a.s. existence of trace for SLE_κ immediately from the above corollary.

Example 2. (non-constant κ) Consider measurable $\kappa : [0, T] \rightarrow [0, \bar{\kappa}]$, with $\bar{\kappa} < 2$, and then

$$U_t = \int_0^t \sqrt{\kappa(s)} dB_s.$$

A.s. existence of trace for “ SLE_κ with non-constant κ ” follows immediately from the above corollary. Remark that for piecewise constant κ , given by (κ_i) on a finite partition of $[0, T]$, this conclusion can also be given by a suitable concatenation argument, relying on a.s. existence of trace for each classical SLE_{κ_i} .

Example 3. (\mathcal{H} -perturbations) Consider, with $h \in \mathcal{H}_T$,

$$U_t = \sqrt{\kappa}B_t + h_t$$

Then $\beta_s := \sqrt{\kappa}(B_t - B_{t-s}) + h_t - h_{t-s}$. The corollary applies with Brownian motion, $N_t = \sqrt{\kappa}(B_t - B_{t-s})$, and deterministic $A_t = h_t - h_{t-s}$. Remark that a.s. existence of trace, for $\kappa > 0$, is also obtained as consequence of existence of trace for classical SLE_κ and the Cameron–Martin theorem. Modifying the example to

$$U_t = \int_0^t \sqrt{\kappa(s)} dB_s + h_t,$$

without imposing a lower positive bound on κ , rules out the Cameron–Martin argument, but Corollary 1 still applies and yields existence of trace a.s.

Example 4. (Ornstein–Uhlenbeck drivers) Consider $U_t = Z_t - Z_0$ where Z is a standard OU process, say with dynamics $dZ = -\lambda Z dt + \sqrt{\lambda} dB_t$ started in its invariant measure. By reversibility of this process, the time-reversed driver β has the same law. Existence of trace (for SLE driven by such OU processes) is then a consequence of Corollary 1.

Corollary 2. Consider

$$U_t = F(t, B_t).$$

with

$$F = F(t, x) \in C^{1,2}, F(0, 0) = 0 \text{ and } |F'(t, x)|^2 \leq \kappa < 2.$$

Assume furthermore, for α large enough (depending only on κ)

$$(1.6) \quad \mathbb{E} \left[\exp \left(\alpha \int_0^T \left\{ \dot{F}(r, B_r) - \frac{1}{2} F''(r, B_r) + F'(r, B_r) \frac{B_r}{r} \right\}^2 dr \right) \right] < \infty.$$

Then the Loewner-trace $\gamma =: \Phi_{SL}(U)$ exists.

Example 5. Fix $p > 0$ and consider, for the sake of argument on $[0, T]$ with $T \leq 1$,

$$U_t = t^p B_t.$$

We insist that there is no “cheap” way to such results. In particular, there is no “comparison result” for SLE that would yield existence of trace based on $t^p \leq 1$ on $[0, 1]$ and existence of trace for SLE_1 , say. (A related question by O. Angel was negatively answered in [LMR10].)

This is a special case of $U_t = F(t, B_t)$. To apply Corollary 2 one needs to check condition (1.6) which boils down to exponential moments for

$$Z_T := \alpha \int_0^T \{r^{p-1} B_r\}^2 dr.$$

with fixed large α , depending on κ . Note that $\mathbb{E}(Z_T) < \infty$. The centered random variable $Z_T^c := Z_T - \mathbb{E}(Z_T)$ lives in the second homogenous chaos over Wiener-space. Exponential moments of Z_T^c are then guaranteed, e.g. by using results from [Led94, Ch. 5], provided that the second moment of Z_T^c is small enough. But this can be achieved by choosing $T > 0$ small enough.

Example 6. Consider

$$U_t = t \log(1 + B_t^2).$$

we leave it to the reader to check that Corollary 2 applies and yields existence of trace on $[0, T]$ with T small enough.

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2. SOME EXACT PRESENTATIONS OF f' .

Lemma 1. For each fixed $t \geq 0$ and $U \in C[0, t]$, define $\beta_s = U_t - U_{t-s}$, $0 \leq s \leq t$. Then

$$(2.1) \quad \log |f'_t(z + U_t)| = \int_0^t \frac{2(X_r^2 - Y_r^2)}{(X_r^2 + Y_r^2)^2} dr$$

where $z = x + iy$ and (X_s, Y_s) , $s \in [0, t]$ is the solution of the ODE

$$(2.2) \quad dX_s = d\beta_s - \frac{2X_s}{X_s^2 + Y_s^2} ds, \quad X_0 = x$$

$$(2.3) \quad dY_s = \frac{2Y_s}{X_s^2 + Y_s^2} ds, \quad Y_0 = y.$$

Moreover, $G_s := \beta_s - X_s$ defines a C^1 -function with,

$$(2.4) \quad \dot{G}_s = \frac{2X_s}{X_s^2 + Y_s^2}.$$

Proof. For each $z \in \mathbb{H}$, the path $g_{t-s}(f_t(z))$ joins z to $f_t(z)$ as s varies from 0 to t . It is then easy to see that

$$f_t(z + U_t) = P_t(z) + U_t$$

where $P_s(z)$ for $s \in [0, t]$ is the solution of ODE

$$\dot{P}_s(z) = \frac{-2}{P_s(z) + \beta_s}, \quad P_0(z) = z$$

Writing in polar form, $P'_s = r_s e^{i\theta_s}$, we see that

$$\operatorname{Re}\left(\frac{|P'_s|}{P'_s} \partial_s P'_s\right) = \operatorname{Re}(e^{-i\theta_s} (e^{i\theta_s} \partial_s r_s + i r_s e^{i\theta_s} \partial_s \theta_s)) = \partial_s r_s$$

So it follows that,

$$\partial_s \log |P'_s| = \operatorname{Re} \left(\frac{1}{P'_s} \partial_s P'_s \right)$$

Noting that $\partial_s P'_s = (\partial_s P_s)'$,

$$\partial_s \log |P'_s| = \operatorname{Re} \left(\frac{1}{P'_s} \left(\frac{-2}{P_s + \beta_s} \right)' \right) = 2 \operatorname{Re} ((P_s + \beta_s)^{-2})$$

$$\implies \log |P'_s| = 2 \int_0^s \operatorname{Re} ((P_r + \beta_r)^{-2}) dr$$

and the claim follows. □

Proposition 1. Fix $t \geq 0$. Let $U \in C^\alpha$ with $\alpha \in (1/3, 1/2]$. With G, β as in the previous lemma,

$$(2.5) \quad \log |f'_t(z + U_t)| = M_t - \int_0^t \dot{G}_r^2 dr + \log\left(\frac{Y_t}{y}\right) - \log\left(\frac{X_t^2 + Y_t^2}{x^2 + y^2}\right)$$

where M_t is given as rough integral

$$(2.6) \quad M_t = \lim_n \sum_{[s,t] \in \pi_n} \dot{G}_s(\beta_t - \beta_s) + \dot{G}'_s \frac{1}{2}(\beta_t - \beta_s)^2$$

with the Gubinelli derivate

$$\dot{G}'_s := \dot{Y}_s / Y_s - \dot{G}_s^2.$$

If in addition, U (equivalently: β , as defined in the previous lemma) has continuous finite quadratic-variation in sense of Föllmer (along π) then

$$(2.7) \quad \log |f'_t(z + U_t)| = M_t^\pi + \frac{1}{2} \int_0^t \dot{G}'_s d[\beta]_s^\pi - \int_0^t \dot{G}_r^2 dr + \log\left(\frac{Y_t}{y}\right) - \log\left(\frac{X_t^2 + Y_t^2}{x^2 + y^2}\right)$$

with (deterministic) Föllmer–Itô integral

$$(2.8) \quad M_t^\pi = \lim_n \sum_{[u,v] \in \pi_n} \dot{G}_u(\beta_v - \beta_u) =: \int_0^t \dot{G} d^\pi \beta.$$

Remark 6. When $U \in \mathcal{H}$, i.e. in the case of finite energy driver, $[\beta] \equiv 0$ and $M \equiv M^\pi$ reduces to a classical Riemann–Stieltjes integral.

Proof. Consider first the case of U (equivalently: β) in C^1 . Then

$$\begin{aligned}
& \dot{G}_r d\beta_r - \frac{1}{2} \dot{G}_r^2 dr + \frac{Y_r dY_r}{X_r^2 + Y_r^2} - \frac{2X_r dX_r + 2Y_r dY_r}{X_r^2 + Y_r^2} \\
&= \frac{2X_r}{X_r^2 + Y_r^2} d\beta_r - \frac{2X_r^2}{(X_r^2 + Y_r^2)^2} dr - \frac{2X_r dX_r}{X_r^2 + Y_r^2} - \frac{Y_r dY_r}{X_r^2 + Y_r^2} \\
&= \frac{2X_r}{X_r^2 + Y_r^2} d(\beta_r - X_r) - \frac{2X_r^2}{(X_r^2 + Y_r^2)^2} dr - \frac{2Y_r^2}{(X_r^2 + Y_r^2)^2} dr \\
&= \frac{4X_r^2}{(X_r^2 + Y_r^2)^2} dr - \frac{2X_r^2}{(X_r^2 + Y_r^2)^2} dr - \frac{2Y_r^2}{(X_r^2 + Y_r^2)^2} dr \\
&= \frac{2(X_r^2 - Y_r^2)}{(X_r^2 + Y_r^2)^2} dr
\end{aligned}$$

Next note that

$$\frac{1}{2} \dot{G}_r^2 dr + \frac{1}{2} \dot{Y}_r^2 dr = \frac{\dot{Y}_r}{Y_r} dr$$

and

$$\frac{Y_r dY_r}{X_r^2 + Y_r^2} = \frac{1}{2} \dot{Y}_r^2 dr = \frac{\dot{Y}_r}{Y_r} dr - \frac{1}{2} \dot{G}_r^2 dr$$

Putting all together, we get

$$\frac{2(X_r^2 - Y_r^2)}{(X_r^2 + Y_r^2)^2} dr = \dot{G}_r d\beta_r - \dot{G}_r^2 dr + \frac{\dot{Y}_r}{Y_r} dr - \frac{2X_r dX_r + 2Y_r dY_r}{X_r^2 + Y_r^2}$$

and integrating both side, the claim follows with $M_t = \int_0^t \dot{G}_s d\beta_s$. In the case of rough driver, meaning U (equivalently: β) in C^α with $\alpha > 1/3$ the idea is to exploit a cancellation between $\dot{G}_r d\beta_r$ and $-\frac{2X_r dX_r}{X_r^2 + Y_r^2}$. We can in fact rely on basic theory of controlled rough path to see existence of the rough integral M_t . It suffices to note that the integrand \dot{G} is controlled by the integrator β . To see this, write

$$\dot{G}_s = \frac{2X_s}{X_s^2 + Y_s^2} =: \varphi(X_s, Y_s)$$

and since φ is smooth and well-defined (as long as $y > 0$ fixed), and Y plainly Lipschitz, it is easy to see that (or just apply directly Exercise 7.8 in [FH14])

$$\dot{G}_s - \dot{G}_r = \partial_x \varphi(X_s, Y_s)(X_s - X_r) + O(|s - r|^{2\alpha}) = \partial_x \varphi(X_s, Y_s)(\beta_s - \beta_r) + O(|s - r|^{2\alpha})$$

which guarantees existence of (2.6) as rough path integral (Theorem 4.10 in [FH14]). The second part concerning the splitting into Itô–Föllmer integral and quadratic variation part, is similar to [FH14, Ch. 5.3], using in particular Lemma 5.9.. \square

3. A DETERMINISTIC ESTIMATE ON f' .

Theorem 4. *In the context of Proposition 1, with continuous finite quadratic-variation in sense of Föllmer so that $d[\beta]_s^\pi/ds \leq \kappa < 2$ one has the following estimate*

$$|f'_t(iy + U_t)| \leq \exp \left[M_t^\pi - \int_0^t \dot{G}_r d(r + \tfrac{1}{2}[\beta]_r) \right]$$

where $\int_0^t \dot{G}_r d^\pi \beta_r = M_t^\pi$ is the Itô–Föllmer type integral introduced in (2.8).

Proof. From (5) and definition for G' , also taking $x = 0$ so that $z = iy$ (i.e. $x = 0$),

$$\begin{aligned} \log |f'_t(iy + U_t)| &= \int_0^t \dot{G}_r d\beta_r - \int_0^t \dot{G}_r^2 dr - \frac{1}{2} \int_0^t \dot{G}_r^2 d[\beta]_r \\ &\quad + \log\left(\frac{Y_t}{y}\right) - \log\left(\frac{X_t^2 + Y_t^2}{y^2}\right) + \frac{1}{2} \int_0^t \frac{\dot{Y}_r}{Y_r} d[\beta]_r \end{aligned}$$

Using positivity of \dot{Y}_r/Y_r ,

$$\begin{aligned} &\log\left(\frac{Y_t}{y}\right) - \log\left(\frac{X_t^2 + Y_t^2}{y^2}\right) + \frac{1}{2} \int_0^t \frac{\dot{Y}_r}{Y_r} d[\beta]_r \\ &\leq \log\left(\frac{Y_t}{y}\right) - 2 \log\left(\frac{Y_t}{y}\right) + \frac{\kappa}{2} \int_0^t \frac{\dot{Y}_r}{Y_r} dr \\ &= \left(\frac{\kappa}{2} - 1\right) \log\left(\frac{Y_t}{y}\right) \\ &\leq 0. \end{aligned}$$

and the desired estimate follows. \square

4. PROOF OF THEOREM 3

As immediate corollary of Theorem 4, noting

$$\int_0^t \dot{G}_r^2 dr \geq \frac{1}{\kappa} \int_0^t \dot{G}_r^2 d[\beta]_r^\pi,$$

we obtain the (still pathwise) estimate

$$(4.1) \quad |f'_t(iy + U_t)| \leq \exp \left[\int_0^t \dot{G}_r d^\pi \beta_r - \left(\frac{1}{2} + \frac{1}{\kappa} \right) \int_0^t \dot{G}_r^2 d[\beta]_r^\pi \right].$$

and then, with $b := 2(\frac{1}{2} + \frac{1}{\kappa})$

$$|f'_t(iy + U_t)|^b \leq \exp \left[b \int_0^t \dot{G}_r d^\pi \beta_r - \frac{b^2}{2} \int_0^t \dot{G}_r^2 d[\beta]_r^\pi \right].$$

Note $b > 2$, a consequence of $\kappa < 2$. We now assume that, for some $A \in \mathcal{H}_t$,

$$\beta = N + A$$

It is then immediate, using $[\beta] = [N]$, that

$$|f'_t(iy + U_t)| \leq \exp \left[\int_0^t \dot{G}_r dN_r^\pi + (*) - \int_0^t \dot{G}_r^2 d(r + \frac{1}{2}[N]_r^\pi) \right]$$

where

$$(*) = \int_0^t \dot{G}_r dA_r = \int_0^t \dot{G}_r \dot{A}_r dr \leq \epsilon \int_0^t \dot{G}_r^2 dr + \frac{1}{4\epsilon} \int_0^t \dot{A}_r^2 dr.$$

As a consequence, arguing exactly like in obtaining (4.1)

$$|f'_t(iy + U_t)| \leq \exp \left[\int_0^t \dot{G}_r dN_r^\pi - \left(\frac{1-\epsilon}{\kappa} + \frac{1}{2} \right) \int_0^t \dot{G}_r^2 d[N]_r \right] \cdot \exp \left[\frac{1}{4\epsilon} \|A\|_t^2 \right].$$

5. FINITE ENERGY DRIVERS, PROOF OF THEOREM 2

We show (i) estimate (1.2), (ii) existence of Loewner-trace γ as a simple curve, (iii) uniform 1/2-Hölder regularity of γ , (iv) Lipschitz continuity of $\gamma(t^2)$, (v) continuity of the Schramm–Loewner in uniform topology, on bounded sets in \mathcal{H}_T and (vi) continuity of the trace in $1 + \epsilon$ -variation topology w.r.t. Cameron-Martin topology on the driver.

(i) The proof of estimate (1.2) is a straight-forward consequence of Theorem 4. Indeed, let U of finite energy on $[0, t]$, note that U and β (with $\beta_s := U_t - U_{t-s}$ here) have zero quadratic variation. Then

$$\log |f'_t(iy + U_t)| \leq M_t - \int_0^t \dot{G}_r^2 dr$$

and conclude with

$$\begin{aligned} M_t &= \int_0^t \dot{G}_r d\beta_r = \int_0^t \dot{G}_r \dot{\beta}_r dr \\ &\leq \int_0^t \dot{G}_r^2 dr + \frac{1}{4} \int_0^t \dot{\beta}_r^2 dr. \end{aligned}$$

In fact, from Proposition 1, since β has zero quadratic variation,

$$\log |f'_t(z + U_t)| = \int_0^t \dot{G}_r d\beta_r - \int_0^t \dot{G}_r^2 dr + \log\left(\frac{Y_t}{y}\right) - \log\left(\frac{X_t^2 + Y_t^2}{x^2 + y^2}\right)$$

and using the same argument as above, we obtain a better bound

$$(5.1) \quad |f'_t(x + iy + U_t)| \leq \frac{y}{Y_t} \left(1 + \frac{x^2}{y^2}\right) \exp\left[\frac{1}{4} \|U\|_t^2\right]$$

which implies $|f'_t(z + U_t)|$ remains bounded if z remains in a cone $\{z | \operatorname{Re}(z) \leq M \operatorname{Im}(z)\}$

(ii) This is clear from part (i) and Lemma 2 in the appendix, where it is shown

$$\gamma_t = \lim_{y \rightarrow 0+} f_t(iy + U_t)$$

exists as a continuous limit. The fact that γ is simple follows e.g. from [L05] or [TRZ13x].

(iii) We show that

$$\|\gamma\|_{\frac{1}{2}} \leq g(\|U\|_T)$$

for some continuous function $g : [0, \infty) \rightarrow (0, \infty)$. (In fact, in the proof below reveals the possible choice $g(x) = Ce^{Cx^2}$.) Define

$$v(t, y) := \int_0^y |f'_t(ir + U_t)| dr$$

Note that,

$$|\gamma(t) - f_t(iy + U_t)| \leq v(t, y)$$

and by an application of Koebe's one-quater Theorem,

$$(5.2) \quad v(t, y) \geq \frac{y}{4} |f'_t(iy + U_t)|$$

In the proof below, we will choose $y = \sqrt{t-s}$. Now,

$$\begin{aligned} |\gamma(t) - \gamma(s)| &\leq |\gamma(t) - f_t(U_t + iy)| \\ &\quad + |\gamma(s) - f_s(U_s + iy)| \\ &\quad + |f_t(U_s + iy) - f_s(U_s + iy)| \\ &\quad + |f_t(U_t + iy) - f_t(U_s + iy)| \end{aligned}$$

The first two terms are bounded by $v(t, y)$ and $v(s, y)$ respectively. For the third term, Lemma 3.5 in [JVL11] and (5.2) implies,

$$|f_t(U_s + iy) - f_s(U_s + iy)| \leq Cv(s, y)$$

For the fourth term,

$$|f_t(U_t + iy) - f_t(U_s + iy)| \leq |U_t - U_s| \sup_{r \in [0,1]} |f'_t(rU_t + (1-r)U_s + iy)|$$

Note that

$$|U_t - U_s| \leq y \|U\|_{\frac{1}{2}}$$

and by Lemma 3.6 in [JVL11], there exist constant C and α such that

$$\begin{aligned} |f'_t(rU_t + (1-r)U_s + iy)| &\leq C \max \left\{ 1, \left(\frac{|U_t - U_s|}{y} \right)^\alpha \right\} |f'_t(iy + U_t)| \\ &\leq C \max \left\{ 1, \|U\|_{\frac{1}{2}}^\alpha \right\} |f'_t(iy + U_t)| \end{aligned}$$

and using (5.2) again,

$$|f_t(U_t + iy) - f_t(U_s + iy)| \leq C \|U\|_{\frac{1}{2}} \max \left\{ 1, \|U\|_{\frac{1}{2}}^\alpha \right\} v(t, y)$$

Finally, from part (i)

$$\begin{aligned} v(t, y) &\leq y \exp \left\{ \frac{1}{4} \|U\|_T^2 \right\} \\ \|U\|_{\frac{1}{2}} &\leq \|U\|_T \end{aligned}$$

giving us

$$|\gamma_t - \gamma_s| \leq \sqrt{t-s} g(\|U\|_T)$$

completing the proof.

(iv) We will use the results from [TRZ13x] for the proof of this part. In particular, we recall from Theorem 3.1 in [TRZ13x] that if $\|U\|_{\frac{1}{2}} < 4$, then there exist a $\sigma, c > 0$ such that for all $y > 0$,

$$(5.3) \quad \sigma\sqrt{t} \leq \text{Im}(f_t(iy + U_t)) \leq \sqrt{y^2 + 4t}$$

and from Lemma 2.1 in [TRZ13x]

$$|\text{Re}(f_t(iy + U_t))| \leq c\sqrt{t}$$

so that trace γ lies inside a cone at 0 and $|f_t(i\sqrt{t} + U_t)| \leq c\sqrt{t}$. We first assume that $\|U\|_{\frac{1}{2}, [0, T]} < 4$. From the proof of part (iii), we have

$$|\gamma_t - \gamma_s| \lesssim v(t, \sqrt{t-s}) + v(s, \sqrt{t-s})$$

If $s, t \geq \epsilon$, using bound 5.1

$$v(t, \sqrt{t-s}) + v(s, \sqrt{t-s}) \lesssim \left(\frac{1}{Y_t} + \frac{1}{Y_s}\right)(t-s) \lesssim \frac{1}{\sqrt{\epsilon}}(t-s)$$

which implies γ is Lipchitz on $[\epsilon, T]$. For proving $|\gamma_{t^2} - \gamma_{s^2}| \lesssim |t-s|$, note that we can assume $s = 0$, for otherwise we can consider the image of γ under conformal map $g_{s^2} - U_{s^2}$ whose derivative of the inverse $f'(\cdot + U_{s^2})$ remains bounded in a cone. Finally again using 5.1 and 5.3,

$$\begin{aligned} |\gamma_{t^2}| &\leq |\gamma_{t^2} - f_{t^2}(it + U_{t^2})| + |f_{t^2}(it + U_{t^2})| \\ &\lesssim v(t^2, t) + t \\ &\lesssim \frac{t^2}{Y_{t^2}} + t \\ &\lesssim t \end{aligned}$$

Finally, if $\|U\|_{\frac{1}{2}, [0, T]} \geq 4$, we split $[0, T] = \cup_{k=0}^{n-1} [\frac{kT}{n}, \frac{(k+1)T}{n}]$ such that for each k , $\|U\|_{\frac{1}{2}, [\frac{kT}{n}, \frac{(k+1)T}{n}]} < 4$. Note again that

$$\gamma[0, T] = \gamma[0, T/n] \cup f_{\frac{T}{n}} g_{\frac{T}{n}}(\gamma[T/n, T])$$

From above, $\gamma(t^2)$ is Lipchitz on $[0, T/n]$. The chain $g_{T/n}(\gamma_{T/n+t}) - U_{T/n}, t \in [0, T/n]$ is driven by $U_{T/n+t} - U_{T/n}$ and since $f'_{T/n}(\cdot + U_{T/n})$ is bounded (from 5.1 and the fact that trace remains in a cone), $\gamma(t^2)$ is also Lipchitz on $[T/n, 2T/n]$. Iterating this argument then completes the proof.

(v) Consider If U^n is a sequence of Cameron-Martin paths with $\|U^n - U\|_\infty \rightarrow 0$ and

$$\sup_n \|U^n\|_T + \|U\|_T < \infty$$

We need to show that for any $\alpha < \frac{1}{2}$,

$$\|\gamma^n - \gamma\|_\alpha \rightarrow 0.$$

We have,

$$\begin{aligned} |\gamma^n(t) - \gamma(t)| &\leq |\gamma^n(t) - f_t^n(iy + U_t^n)| \\ &\quad + |f_t^n(iy + U_t^n) - \gamma(t)| \\ &\quad + |f_t^n(iy + U_t^n) - f_t(iy + U_t)| \end{aligned}$$

Note that for fixed $y > 0$,

$$\lim_{n \rightarrow \infty} |f_t^n(iy + U_t^n) - f_t(iy + U_t)| = 0$$

uniformly in t on compacts. From part (i)

$$\begin{aligned} |\gamma^n(t) - f_t^n(iy + U_t^n)| + |f_t^n(iy + U_t^n) - \gamma(t)| &\leq v^n(t, y) + v(t, y) \\ &\leq y \left(\exp \left\{ \frac{1}{4} \|U^n\|_T^2 \right\} + \exp \left\{ \frac{1}{4} \|U\|_T^2 \right\} \right) \end{aligned}$$

Thus,

$$\lim_{y \rightarrow 0+} |\gamma^n(t) - f_t^n(iy + U_t^n)| + |f_t^n(iy + U_t^n) - \gamma(t)| = 0$$

uniformly in n and t . Since y can be chosen arbitrarily small,

$$\lim_{n \rightarrow \infty} \|\gamma^n - \gamma\|_\infty = 0$$

Finally note that from part (iii)

$$\sup_n \|\gamma^n\|_{\frac{1}{2}} < \infty$$

and standard interpolation argument concludes the proof.

(vi) Let U^n is a sequence with $\|U^n - U\|_T \rightarrow 0$ as $n \rightarrow \infty$. From part (v), we have $\|\gamma^n - \gamma\|_\infty \rightarrow 0$. We claim that $\sup_n \|\gamma^n\|_{1-var} < \infty$, which together with standard interpolation argument implies $\|\gamma^n - \gamma\|_{1+\epsilon-var} \rightarrow 0$ as $n \rightarrow \infty$. From the proof of part (iv), we see that if $\|U\|_{\frac{1}{2}} < 4 - \delta$ (and thus $\|U^n\|_{\frac{1}{2}} < 4 - \delta$ for n large enough), then $\gamma^n(t^2)$ is Lipchitz uniformly in n and thus $\sup_n \|\gamma^n\|_{1-var} < \infty$.

If $\|U\|_{\frac{1}{2}} \geq 4$, we choose a m large enough and dissect $[0, T] = \cup_{k=0}^{m-1} [\frac{kT}{m}, \frac{(k+1)T}{m}]$ such that for all n and $k \leq m-1$, $\|U^n\|_{\frac{1}{2}, [\frac{kT}{m}, \frac{(k+1)T}{m}]} < 4 - \delta$ and similar iteration argument as in proof of part (iv) again implies $\sup_n \|\gamma^n\|_{1-var} < \infty$, completing the proof.

6. PROOF OF COROLLARY 2

We consider Loewner drivers of the form $U_t = F(t, B_t)$ where B is a standard Brownian motion. For a fixed time $t > 0$, the process $\beta_s = \beta_s^t = U_t - U_{t-s}$ is the time reversal of U . Note that $W_s = B_t - B_{t-s}$ is another Brownian motion and a martingale w.r.t. to its natural completed filtration \mathcal{F}_s satisfying usual hypothesis. We recall the following classical result on expansion of filtration. See [Pro90, Ch. 6] for details.

Theorem 5. *Brownian motion W remains a semimartingale w.r.t. expanded filtration $\tilde{\mathcal{F}}_s := \mathcal{F}_s \vee \sigma(W_t) = \mathcal{F}_s \vee \sigma(B_t)$. Moreover,*

$$W_s = \tilde{W}_s + \int_0^s \frac{W_t - W_r}{t-r} dr$$

where \tilde{W} is a Brownian motion adapted to the filtration $\tilde{\mathcal{F}}$.

We now prove that β_s is a semimartingale w.r.t. to filtration $\tilde{\mathcal{F}}$ and provide its explicit decomposition into martingale and bounded variation part. More precisely, we claim

$$\begin{aligned} \beta_s &= \int_0^s F'(t-r, B_{t-r}) dW_r + \int_0^s \left(\dot{F}(t-r, B_{t-r}) - \frac{1}{2} F''(t-r, B_{t-r}) \right) dr \\ &= \int_0^s F'(t-r, B_{t-r}) d\tilde{W}_r \\ &\quad + \int_0^s \left(\dot{F}(t-r, B_{t-r}) - \frac{1}{2} F''(t-r, B_{t-r}) + F'(t-r, B_{t-r}) \frac{B_{t-r}}{t-r} \right) dr \end{aligned}$$

To see this, just use Itô's formula,

$$\beta_s = \int_{t-s}^t F'(r, B_r) dB_r + \int_0^s \left(\dot{F}(t-r, B_{t-r}) + \frac{1}{2} F''(t-r, B_{t-r}) \right) dr$$

and note that by computing the difference between forward (Itô) and backward stochastic integral,

$$\int_{t-s}^t F'(r, B_r) dB_r = \int_0^s F'(t-r, B_{t-r}) dW_r - \int_0^s F''(t-r, B_{t-r}) dr.$$

The proof of corollary 2 is the completed by application of Theorem 3.

Remark 7. If function $F'(t, x)$ is not space dependent, e.g. $F(t, x) = t^p x$ or $F(t, x) = \sqrt{\kappa}x$, we can apply the formula

$$\int_{t-s}^t F'(r) dB_r = \int_0^s F'(t-r) dW_r$$

Note that right-hand side is indeed a martingale w.r.t. the filtration \mathcal{F} and we do not have to work with expanded filtration $\tilde{\mathcal{F}}$. In this case the canonical decomposition of β is given by

$$\beta_s = \int_0^s F'(t-r) dW_r + \int_0^s \dot{F}(t-r, B_{t-r}) dr$$

and Theorem 3 again can be applied considering β as a semimartingale w.r.t. the filtration \mathcal{F} .

APPENDIX

We collect some variations on familiar results concerning existence of trace via moments of f' .

Lemma 2. Suppose there exist a $\theta < 1$ and $y_0 > 0$ such that for all $y \in (0, y_0]$

$$(6.1) \quad \sup_{t \in [0, T]} |f'_t(iy + U_t)| \leq y^{-\theta}$$

then the trace exists.

Proof. Note that for $y_1 < y_2 < y_0$,

$$|f_t(iy_2 + u_t) - f_t(iy_1 + U_t)| = \left| \int_{y_1}^{y_2} f'_t(ir + U_t) dr \right| \leq \int_{y_1}^{y_2} r^{-\theta} dr = \frac{1}{1-\theta} (y_2^{1-\theta} - y_1^{1-\theta})$$

which implies that $f_t(iy + U_t)$ is Cauchy in y and thus

$$\gamma_t = \lim_{y \rightarrow 0+} f_t(iy + U_t)$$

exists. For continuity of γ , observe that

$$|\gamma_t - f_t(iy + U_t)| \leq \frac{y^{1-\theta}}{1-\theta}$$

Now,

$$\begin{aligned} |\gamma_t - \gamma_s| &\leq |\gamma_t - f_t(iy + U_t)| + |\gamma_s - f_s(iy + U_s)| + |f_t(iy + U_t) - f_s(iy + U_s)| \\ &\lesssim y^{1-\theta} + |f_t(iy + U_t) - f_s(iy + U_s)| \end{aligned}$$

It is easy to see that for $y > 0$,

$$\lim_{s \rightarrow t} |f_t(iy + U_t) - f_s(iy + U_s)| = 0$$

and since y was arbitrary, this concludes the proof. \square

Lemma 3. If U is weakly $\frac{1}{2}$ -Holder and there exist constant $b > 2$, $\theta < 1$ and $C < \infty$ such that for all $t \in [0, T]$ and $y > 0$

$$\mathbb{P}[|f'_t(iy + U_t)| \geq y^{-\theta}] \leq Cy^b$$

then the trace exists.

Proof. By using of Borel-Cantelli lemma, it is easy that almost surely for n large enough,

$$|f'_{k2^{-2n}}(i2^{-n} + U_{k2^{-2n}})| \leq 2^{n\theta}$$

for all $k = 0, 1, \dots, 2^{2n} - 1$. Now applying results in section 3 of [JVL11] (Lemma 3.7 and distortion Theorem in particular) completes the proof. \square

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